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Cauchy nets and convergent nets on semilinear topological spaces

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ABSTRACT

One defines three Cauchy-type conditions for nets and some adequate convergence properties on semilinear topological spaces. We characterize Cauchy nets using small sets, we establish some relationships with the above convergences by Cantor-type theorems, and compare them.

1. Introduction

Semilinear spaces are extensions of linear spaces (known also as vector spaces), where the axioms referring to the existence of the inverted element and to the distributivity with respect to the scalar's multiplication are omitted. An important example of semilinear space is given by the family of non-void parts (or closed, bounded, compact, totally bounded or convex parts) of a linear normed space with usual operations with subsets (sum and the multiplication with real scalars). If we endow a semilinear space with a topology for which the two operations are continuous, we obtain a semilinear topological space (s.t.s.).

Some examples of s.t.s. are given by the family of closed bounded parts of a linear normed space with one of following topologies: (upper, lower) Hausdorff topology, lower Vietoris topology or proximal topology. They have many applications in medicine, economics and other.

These notions are introduced by the author in [2] (2002) using the name "almost linear space" which differs of the same-called notion of Godini [6] (1985). In the following, we shall name them semilinear spaces, thus being coherent with the definition of [7, p. 26] (2005). We should be aware that the results of [2] and [3] refer completely to the semilinear spaces.

Now we emphasize some important differences between s.t.s. and linear topological spaces.

The first difference refers to the systems of neighbourhoods of a point x (see [2]): if $\mathcal{V}(0)$ is a fundamental system of neighbourhoods of the origin in a s.t.s. (L,σ), then the family $\mathcal{U}(x) = \{U \subseteq L;\ \exists V \in \mathcal{V}(0)$ such that $x + V \subseteq U\}$ isn't a fundamental system of neighbourhoods for x in the topology σ, but it is a fundamental system of neighbourhoods for x in other topology $\boldsymbol{\sigma}_t$, that we call "the translation of the topology σ".

In this paper we find some new differences between s.t.s. and linear topological spaces: Cauchy conditions.

An important tool in Mathematical Analysis is the used Cauchy nets (or Cauchy sequences). They can be employed in fixed point theorems and many other results. Its definition in linear topological spaces can be equivalently formulated by one of the two conditions: for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ such that for all $i, j \geqslant i_V$ we have $x_i - x_j \in V$, or $x_i \in x_j + V$. In s.t.s. the two above conditions lead us to two different notions. They can be connected with two specific

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types of convergences (Section 3). But we are surprised that if a net is convergent in a s.t.s., it isn't necessarily a Cauchy net. This remark is revealed in Example 4.2.

If we want to characterize the above Cauchy conditions using small sets, we observe that the classical formulation "for every $V \in \mathcal{V}(0)$ there exists a finite set $F_V \subset L$ such that $F_V - F_V \subset V$" correlates to "$x_i - x_j \in V$". But unlike the linear topological spaces case, the assertion "for every $V \in \mathcal{V}(0)$ there exists a finite set $F_V \subset L$ and $x \in F_V$ such that $F_V \subset x + V$" fits with "$x_i \in x_{i_V} + V$, for all $i \geqslant i_V$", not to "$x_i \in x_j + V$ all $i, j \geqslant i_V$". So we obtain a new Cauchy-type condition. We formulate necessary and sufficient conditions for the three Cauchy conditions with small sets in Section 3. Some connections between these Cauchy conditions and different type of convergences are studied in Section 4. For a better understanding of the topological notions the reader can see [9].

2. Terminology and notations

Definition 2.1. Let L be a non-void set. We say that L is a semilinear space if it is endowed with two operations, sum and multiplication with scalars from a field Γ:

$$\text{“}+\text{”}: L \times L \to L$$

and

$$\text{“}\cdot\text{”}: \Gamma \times L \to L$$

which verify the following axioms:

(S1) $(x + y) + z = x + (y + z)$, $\forall x, y, z \in L$;
(S2) there exists an element $0 \in L$ such that $x + 0 = 0 + x = x$, $\forall x \in L$;
(S3) $x + y = y + x$, $\forall x, y \in L$;
(S4) $\lambda(\mu x) = (\lambda\mu)x$, $\forall \lambda, \mu \in \Gamma$, $\forall x \in L$;
(S5) $1 \cdot x = x$, $\forall x \in L$;
(S6) $\lambda(x + y) = \lambda x + \lambda y$, $\forall \lambda \in \Gamma$, $\forall x, y \in L$;
(S7) $0 \cdot x = 0$, $\forall x \in L$.

One easily obtains some simple properties on L:

1. $\lambda \cdot 0 = 0$, for any $\lambda \in \Gamma$.
2. $\lambda x_1 = \lambda x_2$, with $\lambda \in \Gamma \backslash \{0\}$ implies $x_1 = x_2$.
3. For every $\lambda \in \Gamma \backslash \{0\}$ and $x \in L \backslash \{0\}$ we obtain $\lambda x \neq 0$.
4. The zero element (with respect to the sum, given by the axiom (S2)) is unique: if θ is an element of L such that $x + \theta = x$ for all $x \in L$ then $0 + \theta = 0$. But 0 verifies (from (S2)) the equality $\theta + 0 = \theta$, so $\theta = 0$.
5. If x is an invertible element of L, i.e. there exists $x' \in L$ such that $x + x' = 0$, then the inverse element x' is unique: if we suppose that there exists another element $x'' \in L$ with $x + x'' = 0$, summing up x' in the both hands sides, we get $x + x'' + x' = x'$ or, equivalently, $(x + x') + x'' = x'$; so $x'' = x'$.

Definition 2.2. Let $(L, +, \cdot)$ be a semilinear space. A subset $L_1 \subset L$ is called *semilinear subspace* of L if L_1 is itself a semilinear space with respect to the same sum and the same multiplication with scalars as in L.

Remark 2.1. ([7, p. 27]) If $(L, +, \cdot)$ is a semilinear space, then $L_1 \subset L$ is a semilinear subspace if and only if $t \in \Gamma$ and $x, x_1, x_2 \in L_1$ imply $tx \in L_1$ and $x_1 + x_2 \in L_1$.

We define the following subsets of L: sets of invertible elements

$$L_{in} = \left\{x \in L;\ \exists x' \in L \text{ such that } x + x' = 0\right\}$$

and the subset of zero difference

$$L_{dif} = \{x \in L;\ x - x = 0\},$$

where $-x$ is a shortcut for $-x = (-1)x$.

Obviously, L_{in} and L_{dif} are non-empty. These two sets, L_{in} and L_{dif}, are semilinear subspaces of L. Moreover, $L_{dif} \subset L_{in}$.

In the sequel we consider the scalar field $\Gamma = \mathbb{R}$ and we endow it with its usual topology.

Definition 2.3. Consider a topology σ on $(L, +, \cdot)$ such that the operations "$+$" $: (L, \sigma) \times (L, \sigma) \to (L, \sigma)$ and "$\cdot$" $: (\mathbb{R}, |\cdot|) \times (L, \sigma) \to (L, \sigma)$ are both continuous. Then σ is called to be a *semilinear topology* and $(L, +, \cdot, \sigma)$ is a *semilinear topological space* (*s.t.s.*).

We give two examples of semilinear topological spaces:

Example 2.1. Let $(X, \|\cdot\|)$ be a normed vector space under the scalar field $\mathbb{R}$ and d be the metric induced by norm; denote by $\mathcal{P}b(X)$ the family of non-void parts of X which are d-bounded. By $S(x_0, \varepsilon)$ we denote the open ball of centre $x_0 \in X$ and radius $\varepsilon > 0$. We also consider the ε-enlargement of the set $A \subset X$:

$$S_\varepsilon(A) = \{y \in X;\ \text{there exists } a \in A \text{ such that } d(a, y) < \varepsilon\}.$$

On $\mathcal{P}b(X)$ one defines the semi-metric $H_d : \mathcal{P}b(X) \times \mathcal{P}b(X) \to \mathbb{R}$,

$$H_d(A, B) = \max\{e(A, B), e(B, A)\},$$

where

$$e(A, B) = \sup\{d(a, B);\ a \in A\},$$

is the *Hausdorff excess of* A *with respect to* B, and $d(a, B) = \inf\{d(a, b);\ b \in B\}$. *The Hausdorff topology* τ_H is the topology induced by semi-metric H.

Hausdorff topology is also defined on $\mathcal{P}b(X)$ by

$$\tau_H = \tau_H^- \vee \tau_H^+,$$

where a basic neighbourhood of a set $A_0 \in \mathcal{P}(X)$ in each topology is given by:

in τ_H^- (*lower Hausdorff topology*)

$$U_-(A_0, \varepsilon) = \{A \in \mathcal{P}(X);\ A_0 \subseteq S_\varepsilon(A)\}, \quad \text{with } \varepsilon > 0,$$

and in τ_H^+ (*upper Hausdorff topology*)

$$U_+(A_0, \varepsilon) = \{A \in \mathcal{P}(X);\ A \subseteq S_\varepsilon(A_0)\}, \quad \text{with } \varepsilon > 0.$$

On $\mathcal{P}(X)$ we can consider the sum of two sets $A + B = \{a + b;\ a \in A,\ b \in B\}$ and the multiplication with real scalars $\lambda A = \{\lambda a;\ a \in A\}$.

$(\mathcal{P}b(X), +, \cdot)$, with the topology induced by the semi-metric H_d, is a s.t.s. The spaces $(\mathcal{P}b(X), \tau_H^-)$ and $(\mathcal{P}b(X), \tau_H^+)$ are also s.t.s.

Example 2.2. If X is a linear topological space under the scalar field $\mathbb{R}$ and $\mathcal{P}(X)$ is the family of non-void parts of X, then *the lower Vietoris topology* τ_V^- on $\mathcal{P}(X)$ is given by the following subbase:

$$V^- = \{A \in \mathcal{P}(X);\ A \cap V \neq \varnothing\},$$

where V is an open subset of X.

The space $(\mathcal{P}(X), +, \cdot, \tau_V^-)$ is a s.t.s., where "+" and "·" are the operations defined in Example 2.1.

Other examples of s.t.s. can be found in [1] and [2]. They are hyperspacial topological spaces. The main properties of them are studied in [4,5,8,12].

Many properties of linear topological spaces hold in the semilinear topological case. For example, if $\mathcal{V}(0)$ is a fundamental system of neighbourhoods of the origin in a s.t.s. (L, σ) then $\mathcal{V}(0)$ satisfies the conditions (see [2]):

(V0) $0 \in V$ for any $V \in \mathcal{V}(0)$;
(V1) $\forall V_1, V_2 \in \mathcal{V}(0)$, $\exists V_3 \in \mathcal{V}(0)$ such that $V_3 \subset V_1 \cap V_2$;
(V2) $\forall V \in \mathcal{V}(0)$, $\exists V_1 \in \mathcal{V}(0)$ such that $V_1 + V_1 \subset V$;
(V3) $\forall V \in \mathcal{V}(0)$, V is absorbent set, that is, for every $x \in L$ there exists $\lambda > 0$ such that $\lambda x \in V$;
(V4) $\forall V \in \mathcal{V}(0)$, $\exists V_1 \in \mathcal{V}(0)$ such that $\mathcal{E}(V_1) \subset V$, where $\mathcal{E}(V_1)$ is the balanced involving of the set $V_1 \subset L$.

If $A \subset L$ the balanced involving of A can be couched by the formula (valid in linear topological spaces, too) $\mathcal{E}(A) = \bigcup_{|\lambda| \leqslant 1} \lambda A$.

In every s.t.s. we can choose a fundamental system of neighbourhoods of the origin formed only by balanced and open sets.

Also, if $\mathcal{V}(0)$ is a family of subsets of a semilinear space L, which satisfies the conditions (V0)–(V4), then it is a system of basic open neighbourhoods of 0 in some topology on L.

As we comment in the Introduction, an important difference between s.t.s. and the linear topological spaces is the following: the family

$$\mathcal{U}_t(x) = \{U \subset L;\ \exists V \in \mathcal{V}(0) \text{ such that } x + V \subset U\}$$

generates a topology on L, which is different from σ. Let us denote it by σ_t and let call it *the translation of the topology* σ or *the translated topology*. This new topology, σ_t, is not generally a semilinear topology [2, Example 3.2]. But the family $\mathcal{V}(0)$ is a fundamental system of neighbourhoods of origin for the topology σ_t, too. The translated topology σ_t is coarser that σ (since the translations are σ-continuous in the origin, for every $x \in L$ and every $U \in \mathcal{V}_\sigma(x)$ there exists $V \in \mathcal{V}_\sigma(0) = \mathcal{V}(0)$ such that $x + y \in U$ for all $y \in V$, so $x + V \subset U$); in fact, σ_t is the coarsest topology τ on L such that the translations are all τ-continuous in the origin (or, equivalent, the translations are τ-continuous). But the topology σ_t may be different from σ (see [2, Example 3.1]).

In order to characterize the T_1-separation of translated topology we particularize the form of the neighbourhoods and we obtain the following property:

Proposition 2.1. *Let (L, σ) be a semilinear topological space under the scalar field $\mathbb{R}$ and $\mathcal{V}(0)$ a fundamental system of neighbourhoods of the origin. The translated topology σ_t is T_1-separated if and only if $\bigcap_{V \in \mathcal{V}(0)}(a + V) = \{a\}$ for every $a \in L$.*

Particularly, taking $a = 0$ in Proposition 2.1 we get a result which is known from [3].

Proposition 2.2. *Consider (L, σ) a semilinear topological space under the scalar field $\mathbb{R}$ and $\mathcal{V}(0)$ a fundamental system of neighbourhoods of the origin. If the translated topology σ_t is T_1-separated then $\bigcap_{V \in \mathcal{V}(0)} V = \{0\}$.*

Definition 2.4. A s.t.s. (L, σ) is said to be 0-*regular* if the origin has a fundamental system of neighbourhoods formed only by σ-closed sets.

Obviously, if (L, σ) is a regular space, it is also 0-regular.

In this paper we consider both the initial topology σ and its translation σ_t on L.

3. Different type of Cauchy conditions

Let $(L, +, \cdot, \sigma)$ be a semilinear topological space under the scalar field $\mathbb{R}$ and $(x_i)_{i \in I}$ be a net from L (i.e. I is a non-void directed set). We intend to define on L a notion which is similar to Cauchy net (from the linear topological case). In the classical case this notion has two equivalent formulations (see, e.g., [10]). But, on semilinear topological spaces, these assertions lead to two different notions because the sum does not satisfy the axiom of the invertible element. So we define Cauchy nets in difference and translated Cauchy nets.

Definition 3.1. By the *family of the sections* of the net $(x_i)_{i \in I}$ we mean the collection $\mathcal{S} = \{S_i;\ i \in I\}$, where $S_i = \{x_j;\ j \geqslant i\}$.

We recall that on the linear topological spaces, a net is a Cauchy net iff the family of their sections contains small sets. Like above, the families which contain small sets can be defined in two equivalent ways. One of them matches the Cauchy nets in difference, but the other one leads us to a new type of Cauchy nets: we call them "semi-Cauchy nets". So we have three different notions if we work in the semilinear case.

Also one can define two significant types of convergences, corresponding to Cauchy nets in difference and translated Cauchy nets.

Remark 3.1. If the net $(x_i)_{i \in I}$ is convergent in difference does not result, generally, that it is a Cauchy net in difference, too. Also the convergence in the translated topology does not imply the translated Cauchy condition.

This situation is unexpected, but a similar behaviour we can meet, for example, in [11], where the convergence and Cauchy condition are defined using an almost metric.

We establish relationships between these convergences and the three Cauchy-type conditions. We will show some characteristic theorems in Cantor way (by using the families of closed sets with finite intersection property and non-void intersection).

Definition 3.2. A collection $\mathcal{F}$ of subsets of a topological space L has the finite intersection property if any finite collection $F_1, F_2, \ldots, F_n \in \mathcal{F}$ has non-empty intersection: $\bigcap_{k=1}^{n} F_k \neq \varnothing$.

Let present these notions below. We omit some proofs when they are simple or they are similar to the classical case.

3.1. Cauchy nets in difference

Definition 3.3. $(x_i)_{i\in I}$ is called *a Cauchy net in difference* if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ such that $x_i - x_j \in V$ for all $i, j \geqslant i_V$.

Definition 3.4. Consider $\mathcal{F}$ a family of non-void parts of L. The family $\mathcal{F}$ contains *small sets in difference* if:

for every $V \in \mathcal{V}(0)$ there exists $F_V \in \mathcal{F}$ such that $F_V - F_V \subset V$.

In the sequel we intend to establish a relationship between the above Cauchy type nets and the property of small sets of the families of their sections.

Proposition 3.1. *Let $(x_i)_{i\in I}$ be a net from L and $\mathcal{S} = \{S_i;\ i \in I\}$ the family of its sections. Then $(x_i)_{i\in I}$ is a Cauchy net in difference if and only if $\mathcal{S}$ contains small sets in difference.*

Definition 3.5. $(x_i)_{i\in I}$ is called *a convergent in difference net* to x if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ having the property $x_i - x \in V$ for all $i \geqslant i_V$.

Remark 3.2. (1) The convergence in difference doesn't imply the Cauchy condition in difference. But, obviously, the following assertion is accomplished: if $(x_i)_{i\in I}$ is a net from a semilinear topological space L which is convergent in difference to $x \in L_{dif}$, then $(x_i)_{i\in I}$ is a Cauchy net in difference.

(2) Let $(x_i)_{i\in I} \subset L$ be a net and $\mathcal{S} = \{S_i;\ i \in I\}$ the family of their sections (where $S_i = \{x_j;\ j \geqslant i\}$). Consider the family of the σ-closure of their sets $\mathcal{S}_1 = \{\overline{S}^{\sigma};\ S \in \mathcal{S}\}$. Then $\mathcal{S}$ and $\mathcal{S}_1$ have the finite intersection property: let $S_1, S_2, \ldots, S_n \in \mathcal{S}$. Then there exists $S \in \mathcal{S}$ such that $S \subset S_k$, $k = \overline{1,n}$ (because the family $\mathcal{S}$ is a directed set). So $\bigcap_{k=1}^{n} S_k \neq \varnothing$. Moreover $\overline{S} \subset \overline{S_k}$, for every $k = \overline{1,n}$, which implies that $\bigcap_{k=1}^{n} \overline{S_k}^{\sigma} \neq \varnothing$.

Theorem 3.1. *Let (L, σ) be a semilinear topological space which is 0-regular. We assume that the translated topology σ_t is T_1 separated. The Cauchy nets in difference from L are convergent in difference if and only if any family $\mathcal{F}$ of σ-closed sets of L which contains small sets in difference and which has the finite intersection property, has non-void intersection.*

Proof. Since (L, σ) is 0-regular, we can work with a fundamental system of σ-closed neighbourhoods of the origin. Let it denote by $\mathcal{V}(0)$.

Consider the family $\mathcal{F}_1$ of all finite intersection of sets of $\mathcal{F}$. The new family $\mathcal{F}_1$ has, obviously, the finite intersection property.

$\mathcal{F}_1$ contains small sets in difference, too.

The family $\mathcal{F}_1$ is ordered by the inclusion: $F_1 \geqslant F_2$ iff $F_1 \subset F_2$.

For any $F \in \mathcal{F}_1$ there exists $x_F \in F$ because $\mathcal{F}_1$ has the finite intersection property. We have obtained the net $(x_F)_{F\in\mathcal{F}_1}$.

This net is a Cauchy net in difference: let $V \in \mathcal{V}(0)$ be an arbitrary neighbourhood; since $\mathcal{F}_1$ contains small sets in difference it follows that there exists $F_V \in \mathcal{F}_1$ such that $F_V - F_V \subset V$. Then for every $F, E \geqslant F_V$ we have $x_F - x_E \in F_V - F_V \subset V$.

From hypotheses $(x_F)_{F\in\mathcal{F}_1}$ is a convergent net in difference, say to x. We will show that $x \in \bigcap_{F\in\mathcal{F}_1} F$.

In order for this, consider an arbitrary $F \in \mathcal{F}_1$. The set

$$S_F = \{x_E;\ E \geqslant F,\ E \in \mathcal{F}_1\}$$

is its section. Because $E \subset F$ and $x_E \in E$ it results that $x_E \in F$, so $S_F \subset F$.

The net $(x_F)_{F\in\mathcal{F}_1}$ is convergent to x, so for every $V \in \mathcal{V}(0)$ there exists $F'_V \in \mathcal{F}_1$ such that $S_F - x \subset V$, for all $F \geqslant F'_V$. We can consider that $F_V = F'_V$. Taking the closure (in the sense of topology σ) of sets in the last inclusion we obtain $\overline{S_{F_V} - x}^{\sigma} \subset V$. So

$$\overline{S_F}^{\sigma} - x \subset \overline{S_{F_V} - x}^{\sigma} \subset V, \quad \forall F \geqslant F_V.$$

We deduce

$$\bigcap_{F\in\mathcal{F}_1} \overline{S_F}^{\sigma} - x \subset \bigcap_{V\in\mathcal{V}(0)} V.$$

But (L, σ_t) is T_1-separated, so $\bigcap_{F\in\mathcal{F}_1} \overline{S_F}^{\sigma} - x = \{0\}$. This means that $\bigcap_{F\in\mathcal{F}_1} \overline{S_F}^{\sigma} = (-x)'$, where $(-x)'$ is the inverse (unique, see p. 3) of the element $-x$. So $(-x)' \in \bigcap_{F\in\mathcal{F}_1} \overline{F}^{\sigma} = \bigcap_{F\in\mathcal{F}_1} F$. It implies that $\bigcap_{F\in\mathcal{F}_1} F \neq \varnothing$, and, by consequently, $\bigcap_{F\in\mathcal{F}} F \neq \varnothing$.

For the reverse assertion, let $(x_i)_{i\in I} \subset L$ be a Cauchy net in difference. Consider the family of its sections $\mathcal{S} = \{S_i;\ i \in I\}$ and the family of its σ-closure $\mathcal{S}_1 = \{\overline{S}^\sigma;\ S \in \mathcal{S}\}$. $\mathcal{S}_1$ satisfies the hypotheses of the theorem:

1. $\mathcal{S}_1$ has the finite intersection property (Remark 3.2).
2. $\mathcal{S}_1$ has small sets in difference: $(x_i)_{i\in I}$ is Cauchy net in difference and use Proposition 3.1. Then $\overline{S_{i_V}}^\sigma - \overline{S_{i_V}}^\sigma \subset \overline{S_{i_V} - S_{i_V}}^\sigma \subset \overline{V}^\sigma = V$.

From hypotheses, the family $\mathcal{S}_1$ has non-void intersection; let $x \in \bigcap_{S\in\mathcal{S}} \overline{S}^\sigma = \bigcap_{i\in I} \overline{S_i}^\sigma$.
It follows that $\overline{S_{i_V}}^\sigma - x \subset \overline{S_{i_V}}^\sigma - \overline{S_{i_V}}^{gs} \subset V$. So for every $i \geqslant i_V$ we have $x_i - x \in V$. $\square$

3.2. Translated Cauchy nets and semi-Cauchy nets

Definition 3.6. $(x_i)_{i\in I}$ is called *a translated Cauchy net* if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ such that $x_i \in x_j + V$ for all $i, j \geqslant i_V$.

Now we find a characteristic property for the family of the sections of the translated Cauchy nets.

Definition 3.7. Let I be a directed set and $\mathcal{F} = (F_i)_{i\in I}$ a decreasing family: for every $i_1 \geqslant i_2$ it follows that $F_{i_1} \subset F_{i_2}$.
The family $\mathcal{F} = (F_i)_{i\in I}$ contains *small translated sets* if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ such that $F_{i_V} \subset F_i + V$, for all $i \geqslant i_V$.

Proposition 3.2. *Consider a semilinear topological space. The family of the sections of a net contains small translated sets if and only if it is a translated Cauchy net.*

Definition 3.8. $(x_i)_{i\in I}$ is called *a convergent net in translated topology* to x if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ having the property $x_i \in x + V$ for all $i \geqslant i_V$.

In the case of translated Cauchy nets a similar result to Theorem 3.1 is the following (well known for any convergent net):

Proposition 3.3. *Let (L, σ) be a semilinear topological space and let σ_t be its translated topology. If a net $(x_i)_{i\in I}$ from L is convergent in the translated topology then the family of the σ_t-closure of their sections has non-void intersection.*

Now consider $\mathcal{F}$ a family of non-void parts of L.

Definition 3.9. The family $\mathcal{F}$ contains *semi-small sets* if:

for every $V \in \mathcal{V}(0)$ there exist $F_V \in \mathcal{F}$ and $x \in F_V$ such that $F_V \subset x + V$.

We expect that the translated Cauchy nets be characterized by the families of the sections which contain semi-small sets. This property does not occur. But the families of the sections which contain semi-small sets are related with another notion:

Definition 3.10. A net $(x_i)_{i\in I}$ is called a *semi-Cauchy net* if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ such that $x_i \in x_{i_V} + V$, for all $i \geqslant i_V$.

Proposition 3.4. *The family of sections of a net contains semi-small sets if and only if the net is semi-Cauchy.*

4. Links between different types of Cauchy nets and convergent nets

The three definitions above of Cauchy nets are not equivalent on L.
The relationship between the translated Cauchy nets and semi-Cauchy nets is obvious:

Remark 4.1. Every translated Cauchy net is a semi-Cauchy net.

In the sequel we establish relationships between the translated Cauchy nets and Cauchy nets in difference.

For example, if $(x_i)_{i\in I}$ is a convergent net in difference to $x \in L_{dif}$, then $(x_i)_{i\in I}$ is a translated Cauchy net.

The following notion (which is trivial on the linear topological spaces) appears in a natural way. It tries to compensate the missing property $x - x = 0$ for $x \in L$.

Definition 4.1. $(x_i)_{i\in I}$ contains *small autodifferences* if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ such that $x_i - x_i \in V$ for all $i \geqslant i_V$.

Every Cauchy net in difference has small autodifferences.

This notion is consistent and it is different from definition of Cauchy net in difference. We can see these in the following example:

Example 4.1. Consider the linear normed space l_2 endowed with the usual norm. Like in Example 2.1 we define on the family $\mathcal{K}(l_2)$ of compact subsets of l_2 the sum and multiplication with real scalars. We endow $\mathcal{K}(l_2)$ with the upper Hausdorff topology τ_H^+ defined in Example 2.1. Let be the sets $A_n = [(1-\frac{1}{n})e_n, (1+\frac{1}{n})e_n]$ in $(\mathcal{K}(l_2), \tau_H^+)$, where $e_n = (0, 0, \dots, 1, 0, \dots)$ is the vector of rank $n \in \mathbb{N}$ from the canonical base of l_2.

The basic neighbourhood of the origin $\{0\}$ in the topology τ_H^+ is given by $U_+(\{0\}, \varepsilon) = \{A \in \mathcal{K}(l_2);\ A \subset S(0, \varepsilon)\}$ where $S(0, \varepsilon)$ is the open ball of centre 0 and radius $\varepsilon > 0$.

▲ If we denote $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ then the sequence $(A_n)_{n\in\mathbb{N}^*}$ contains small autodifferences: $\forall\varepsilon > 0$, $A_n - A_n \in U_+(\{0\}, \varepsilon)$ for n large enough, because

$$\sup\bigl\{d(a, 0); a \in A_n - A_n\bigr\} = \sup\left\{|\lambda - \mu|; \lambda, \mu \in \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]\right\} = \frac{2}{n}.$$

▲ The sequence $(A_n)_{n\in\mathbb{N}^*}$ isn't Cauchy net in difference: if for an arbitrary $\varepsilon > 0$ we have $A_n - A_m \in U_+(\{0\}, \varepsilon)$, then $\sqrt{(1+\frac{1}{n})^2 + (1+\frac{1}{m})^2} < \varepsilon$, for $n \neq m$, in contradiction with $\sqrt{(1+\frac{1}{n})^2 + (1+\frac{1}{m})^2} \geqslant \sqrt{2}$.

In order to indicate a relationship between semi-Cauchy nets and Cauchy nets in difference we introduce the notion below:

Definition 4.2. The family $\mathcal{S} = \{S_i;\ i \in I\}$ of the sections of a net contains *semi-small sets with small translated autodifferences* if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ with the properties

$$\exists x \in S_{i_V} \quad \text{such that} \quad S_{i_V} \subset x + V \tag{1}$$

and

$$x_i - x_i \in V \quad \text{for all } i \geqslant i_V. \tag{2}$$

The following result holds:

Proposition 4.1. *Let (L, σ) be a s.t.s. If $(x_i)_{i\in I} \subset L$ is a translated Cauchy net and the family of its sections contains small sets with small translated autodifferences then $(x_i)_{i\in I}$ is a Cauchy net in difference.*

Proof. According to Proposition 3.4 it is sufficiently to show that, if the family of the sections of a net $(x_i)_{i\in I}$ contains semi-small sets, then $(x_i)_{i\in I}$ is a semi-Cauchy net.

Let $U \in \mathcal{V}(0)$ be an arbitrary neighbourhood of the origin. Then (from (V2)) there exists $V \in \mathcal{V}(0)$ such that

$$V + V + V \subset U. \tag{3}$$

Using the assertion (V4), the set V can be considered as a balanced neighbourhood.

From condition (1) it follows that there exists $j_V \geqslant i_V$ such that $x = x_{j_V}$. So

$$x_i \in x_{j_V} + V, \quad \forall i \geqslant j_V. \tag{4}$$

We fix an arbitrary $j \geqslant j_V$. From (4) we deduce that

$$-x_j \in -x_{j_V} - V. \tag{5}$$

By summing up properly the two hands of the relations (4) and (5) we obtain

$$x_i - x_j \in x_{j_V} - x_{j_V} + V - V. \tag{6}$$

Taking $i = j_V$ in (2) we find $x_{j_V} - x_{j_V} \in V$. So from the conditions (6) and (3) we deduce that $(x_i)_{i\in I}$ is a Cauchy net in difference (finally we use that V is a balanced set). □

Suppose that $(x_i)_{i\in I}$ is a net. We can introduce the following flexible definition:

Definition 4.3. A net $(x_i)_{i\in I}$ from a s.t.s. is called *σ_t-reverse convergent net* to x if:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ having the property $x \in x_i + V$ for all $i \geqslant i_V$.

The links between translated Cauchy nets, σ_t-reverse convergent nets and convergent nets in translated topology are given by the following three results:

Proposition 4.2. *Let be a semilinear topological space for which the translated topology σ_t is T_1-separated. If a net $(x_i)_{i\in I}$ is convergent in the translated topology to x and it is also σ_t-reverse convergent to y then $x = y$.*

Proof. Let $U \in \mathcal{V}(0)$ be an arbitrary neighbourhood of the origin and $V \in \mathcal{V}(0)$ such that $V + V \subset U$. Since I is a directed set we can find an index $i_V \in I$ such that

$$x_i \in x + V \quad \text{and} \quad y \in x_i + V, \quad \forall i \geqslant i_V. \tag{7}$$

So $y \in x + V + V \subset x + U$, for any $U \in \mathcal{V}(0)$. Thus $y \in \bigcap_{U\in\mathcal{V}(0)}(x + U) = \{x\}$ from Proposition 2.1, so $y = x$. □

Proposition 4.3. *Let be a semilinear topological space L for which the translated topology σ_t is T_1-separated. If the net $(x_i)_{i\in I}$ is σ_t-reverse convergent and it is convergent in the translated topology, then it is a translated Cauchy net.*

Proof. Let $U \in \mathcal{V}(0)$ be an arbitrary neighbourhood of the origin and $V \in \mathcal{V}(0)$ such that $V + V \subset U$.

From Proposition 4.2 there exists $x \in L$ such that $(x_i)_{i\in I}$ is convergent to x in the translated topology and $(x_i)_{i\in I}$ is σ_t-revers convergent to x. Consider $i_V \in I$ from the relation (7). It results that

$$x_i \in x + V \in x_j + V + V \subset x_j + U,$$

for any $i, j \geqslant i_U = i_V$. □

Now we want to investigate when a net convergent in translated topology is also σ_t-reverse convergent. The result is more restrictive: if we consider the translated topology σ_t on a semilinear topological space L, the property

$$\overline{x + V}^{\sigma_t} = x + \overline{V}^{\sigma}, \tag{8}$$

is not generally true, where V is an arbitrary neighbourhood of the origin (the translations of σ-closed sets generally do not need to be σ_t-closed).

We indicate two type sufficient conditions which assure the relation (8) (one with respect to the element x; another with respect to the neighbourhood V) when L is a semilinear topological space:

Lemma 4.1. *Let L be a s.t.s. endowed with the translated topology σ_t, which is T_2-separated.*

1. *If L_{in} is the semilinear subspace of invertible elements of L then, for any σ-closed neighbourhood of the origin $V \in \mathcal{V}(0)$ and any $x \in L_{in}$, the set $x + V$ is closed in the translated topology.*
2. *If V is σ_t-compact set then $x + V$ is σ_t-compact, for any $x \in L$.*

Proof. 1. Let $(y_i)_{i\in I}$ be a net from $x + V$ with $y_i \xrightarrow{\sigma_t} y$. If $(x_i)_{i\in I} \subset V$ is a net such that $y_i = x + x_i$ then $x_i = y_i + x'$, where x' is the inverse element of x. Since the translations are σ_t-continuous, the net $(x_i)_{i\in I}$ is σ_t-convergent to an element z, where $z \in \overline{V}^{\sigma} = V$. It follows that $y_i \xrightarrow{\sigma_t} x + z$, so $y = x + z$ (because (L, σ_t) is T_2-separated). This shows that $x + V$ is σ_t-closed.

2. The translations are σ_t-continuous and continuous images of compact sets are compact. □

We need to introduce the following definition:

Definition 4.4. Say that the net $(x_i)_{i\in I}$ from a s.t.s. (L,σ) is *closed-translated* if:

$$\overline{x_i+V}^{\sigma_t}=x_i+\overline{V}^{\sigma},\quad \text{for all } V\in\mathcal{V}(0) \text{ and all } i\in I, \tag{9}$$

where σ_t is the translated topology of σ.

Remark 4.2. From Lemma 4.1 we see that every net from the subspace L_{in} is a closed translated net.

Proposition 4.4. *Let (L,σ) be a semilinear topological space which is 0-regular and let σ_t be its translated topology. Consider $(x_i)_{i\in I}$ a closed-translated net which is also a translated Cauchy net. If $(x_i)_{i\in I}$ is convergent in the translated topology to an element x then it is σ_t-reverse convergent to x.*

Proof. Consider $\mathcal{V}(0)$ a fundamental system of σ-closed neighbourhoods of the origin. For any $V\in\mathcal{V}(0)$ we can find an index $i_V^1\in I$ such that for all $j\geqslant i_V^1$ we have $S_{i_v}\subset x_j+V$, so for all $i,j\geqslant i_V^1$ we get $\overline{S_i}^{\sigma_t}\subset\overline{x_j+V}^{\sigma_t}$. Since $(x_i)_{i\in I}$ is a closed-translated net and (L,σ) is 0-regular we deduce that $\overline{x_j+V}^{\sigma_t}=x_j+V$ for any $j\in I$. Suppose that $(x_i)_{i\in I}$ is σ_t-convergent to x.

For any $V\in\mathcal{V}(0)$ there exists $i_V^2\in I$ such that $x_i\in x+V$, when $i\geqslant i_V^2$, so $S_i\subset x+V$.

Obtain that for every neighbourhood $W=x+V$ of x (in the translated topology), $W\cap S_i\neq\varnothing$. It results that $x\in\overline{S_i}^{\sigma_t}$, for any $i\geqslant i_V$, where $i_V\geqslant i_V^k$, $k=1,2$. Thus $x\in\bigcap_{i\geqslant i_V}\overline{S_i}^{\sigma_t}\subset x_j+V$, for all $j\geqslant i_V$. $\square$

Finally we give an example of net which is: a Cauchy net in difference, a convergent net in difference, a semi-Cauchy net, a convergent net in translated topology, but it isn't a translated Cauchy net.

Example 4.2. Let $e_1,e_2,\dots,e_n,\dots$ be the vectors of canonical base in l_2 and $A_n=\mathcal{I}(e_1,e_2,\dots,e_n)$, where by $\mathcal{I}(e_1,\dots,e_n)$ we denote the set $\mathcal{I}(e_1,e_2,\dots,e_n)=\{\lambda_1e_1+\dots+\lambda_ne_n;\ \lambda_1,\dots,\lambda_n\in\mathbb{R}\}$.

On the family $\mathcal{P}(l_2)$ of non-void subsets of l_2 we consider the topology τ_V^- (see Example 2.2); a neighbourhood of $A_0\in l_2$ in τ_V^- is given by $U=U_1^-\cap\dots\cap U_p^-$, where U_k are open sets such that $A_0\cap U_k\neq\varnothing$, $k=\overline{1,p}$.

Let be the sets $A_0=l_2$ and $\widetilde{A}=\{0\}$. We also make the notation $\mathbb{N}^*=\mathbb{N}\backslash\{0\}$. Then:

- ▲ $A_n\xrightarrow{\tau_V^-}A_0$: let $(U_k)_{k=\overline{1,p}}$, $p\in\mathbb{N}^*$ be an arbitrary non-empty open set from l_2 with $A_0\cap U_k\neq\varnothing$ for all $k=\overline{1,p}$. Consider $x_k\in U_k$ and $\rho_k>0$ such that $S(x_k,\rho_k)\subset U_k$ for all $k=\overline{1,p}$.
  If $x_k=\lambda_1^ke_1+\lambda_2^ke_2+\dots+\lambda_n^ke_n+\cdots$, we choose $y_k=\lambda_1^ke_1+\lambda_2^ke_2+\dots+\lambda_n^ke_n\in A_n$. Then $\|x_k-y_k\|=\|\lambda_{n+1}^ke_{n+1}+\lambda_{n+2}^ke_{n+2}+\cdots\|=\sum_{m=n+1}^{\infty}|\lambda_m^k|^2<\rho_k$ for n sufficiently large, so $A_n\cap S(x_k,\rho_k)\neq\varnothing$, i.e. $A_n\cap U_k\neq\varnothing$.
- ▲ We denote by τ_t the translated topology of τ_V^-. We show that $A_n\overset{\tau_t}{\nrightarrow}A_0$:
  Consider some arbitrary open sets $U_1,\dots,U_p$ with $0\in U_1\cap\dots\cap U_p$. We denote $U=U_1^-\cap\dots\cap U_p^-$. If $B\in U$ then $B\cap U_k\neq\varnothing$, for all $k=\overline{1,p}$.
  Suppose that $A_n\xrightarrow{\tau_t}A_0$: then there exist $n_U\in\mathbb{N}$ and $B\in U$ such that $A_n=A_0+B$ for all $n\geqslant n_U$, a contradiction, because $A_0+B=l_2$.
- ▲ However (A_n) is convergent to $\widetilde{A}$ in translated topology: we can choose $n_U=1$ and $B=\mathcal{I}(e_1,e_2,\dots,e_n)\in U=U_1^-\cap\dots\cap U_p^-$ (where $U_1,\dots,U_p$ are open sets with $0\in U_1\cap\dots\cap U_p$) such that $A_n=\widetilde{A}+B$ for $n\geqslant1$.
- ▲ (A_n) isn't a translated Cauchy net:
  Suppose by contrary that for any open set U (i.e. for any open sets U_k with $0\in U_k$, $k=\overline{1,p}$), there exist $n_U\in\mathbb{N}$ and $B\in\bigcap_{k=\overline{1,p}}U_k^-(\{0\})$ such that $A_n=A_m+B$ for all $n,m\geqslant n_U$. Particularly, for $n<m$, this means that $\mathcal{I}(e_1,e_2,\dots,e_n)=\mathcal{I}(e_1,e_2,\dots,e_n,\dots,e_m)+B$, a contradiction.
- ▲ But we can choose $n_U=m$ and $B=\mathcal{I}(e_1,e_2,\dots,e_m,\dots,e_n)\in\bigcap_{k=\overline{1,p}}U_k^-(\{0\})$ such that, if $n\geqslant m$, the equality $\mathcal{I}(e_1,e_2,\dots,e_m,\dots,e_n)=\mathcal{I}(e_1,e_2,\dots,e_n)+B$ holds (where U_k are arbitrary open sets with $0\in U_k$, for any $k=\overline{1,p}$). Deduce that (A_n) is a semi-Cauchy net.
- ▲ (A_n) is a Cauchy net in difference: for any open sets U_k with $0\in U_k$, $k=\overline{1,p}$, we have $A_n-A_m=A_{\max\{n,m\}}\in\bigcap_{k=\overline{1,p}}U_k^-$ for all $n,m\geqslant1$.
- ▲ (A_n) is convergent in difference to $A_0=l_2$: for any open sets U_k with $0\in U_k$, $k=\overline{1,p}$, we have $A_n-A_0=A_0\in\bigcap_{k=\overline{1,p}}U_k^-$ for all $n,m\geqslant1$.

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References

[1] G. Apreutesei, Hausdorff topology and some operations with subsets, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. 44 (1998) 445–454.
[2] G. Apreutesei, The hyperspacial topologies and almost linear spaces, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. 48 (2002) 3–18.
[3] G. Apreutesei, N. Mastorakis, A. Croitoru, A. Gavrilut, On the translation of an almost linear topology, WSEAS Transactions on Mathematics 8 (9) (2009) 479–488.
[4] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer Academic Publisher, 1993.
[5] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, vol. 580, Springer Verlag, Berlin, 1977.
[6] G. Godini, A framework for best simultaneous approximation: Normed almost linear spaces, J. Approx. Theory 43 (1985) 338–358.
[7] A. Hamel, Variational principles on metric and uniform spaces, Habilitation thesis, Martin-Luther University, Haale, 2005.
[8] E. Klein, A. Thompson, Theory of Correspondences, Willey, Toronto, 1984.
[9] K. Kuratowski, Topology, Academic Press, New York, 1966.
[10] T. Precupanu, Linear Topological Spaces and Some Elements of Convex Analysis, Ed. Acad. Rom., Bucharest, 1992.
[11] M. Turinici, Functional versions of the Caristi–Kirk theorem, Revista de la Union Matematica Argentina 50 (2009) 87–97.
[12] Y. Sonntag, C. Zalinescu, Set convergence: A survey and a classification, Set-Valued Analysis 2 (1994) 339–356.